On the duality relation for correlation functions of the Potts model

Wentao T. Lu and F. Y. Wu

Department of Physics

Northeastern University, Boston, Massachusetts 02115

Abstract

We prove a recent conjecture on the duality relation for correlation functions of the Potts model for boundary spins of a planar lattice. Specifically, we deduce the explicit expression for the duality of the n-site correlation functions, and establish sum rule identities in the form of the Möbius inversion of a partially ordered set. The strategy of the proof is by first formulating the problem for the more general chiral Potts model. The extension of our consideration to the many-component Potts models is also given.

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I. INTRODUCTION

The duality relation for the q-state Potts model [1,2] is an identity [3] relating the partition functions of a Potts model on a planar lattice with that of its dual. Very recently, this duality consideration has been extended to the Potts correlation functions [4,5]. Specifically, it was established that certain duality relations exist for correlation functions of n Potts spins on the boundary of a planar lattice. Explicit expressions for the duality relation have been obtained for $n = 2, 3$ [4] and $n = 4$ [5]; the expression for general n has also been conjectured [5]. It has also been shown that there exist certain sum rule identities, but the explicit form of the identities was not given [5]. The purpose of this paper is the following: We first extend the consideration to the chiral Potts model [6] and obtain its duality relation in a very general form. We then show that this formulation permits us to establish the conjecture of [5]. In addition, it also leads to an expression of the sum rule identities in the form of the Möbius inversion of a partially ordered set. Furthermore, starting from a multi-component chiral Potts model [7], we extend considerations to the multi-component Potts model, which includes the Ashkin-Teller model [9] and its generalization [10]. This extends results reported elsewhere [11] for the 2-component Potts model.

II. THE CHIRAL POTTS MODEL

Consider a q-state spin system on a two-dimensional lattice or, more generally, any planar graph $\mathcal{L}$. Let $x_i = 1, 2, \cdots, q$ denote the spin state of the i-th site and $-J(x_i, x_j)$ the interaction energy between sites i and j, which can be edge-dependent. The interaction is chiral if

$$J(x, x') \neq J(x', x). \tag{1}$$

The partition function of this spin system is

$$Z(u) = \sum_{x_i=1}^{q} \prod_{<ij>} U(x_i, x_j), \tag{2}$$

where $U(x, x') = \exp[J(x, x')/kT]$ and the product is taken over all edges of $\mathcal{L}$.

For our purposes we consider the chiral Potts model [6] for which the Boltzmann factor is cyclic, namely, it satisfies

$$U(x, x') = u(x - x'), \qquad (\text{mod} \quad q). \tag{3}$$

We shall refer to the $q \times q$ matrix $\mathbf{U}$ as the interaction matrix. The chiral Potts model reduces to the standard Potts model upon taking $J(x, x')/kT = K\delta(x, x')$, where δ is the Kronecker delta function which we define as

$$\begin{aligned} \delta(x, x') &= 1, & x = x' \pmod{q} \\ &= 0, & x \neq x' \pmod{q}. \end{aligned} \tag{4}$$

This leads to

$$u(x) = 1 + (e^K - 1)\delta(x, 0). \tag{5}$$

Duality relation for the partition function: The partition function of the chiral Potts model possesses a duality relation [3]. Let $\mathcal{L}^D$ be the dual of $\mathcal{L}$, and let $Z^{(D)}(u^*)$ be the partition function of the chiral Potts model on $\mathcal{L}^D$ with the Boltzmann factor[1]

$$u^*(y) = \frac{1}{\sqrt{q}} \sum_{x=1}^{q} \omega^{xy} u(x), \qquad y = 1, 2, \cdots, q \tag{6}$$

where $\omega = e^{2\pi i/q}$. Then, the duality relation reads [3]

$$q^{-N/2} Z[u(x)] = q^{-N^*/2} Z^{(D)}[u^*(y)], \tag{7}$$

where N and N^* are, respectively, the numbers of sites of $\mathcal{L}$ and $\mathcal{L}^D$. They are related to the number of edges E by the Euler relation $N + N^* = E + 2$. This duality relation holds for arbitrary edge-dependent interactions. Note that $\sqrt{q}u^*(y)$, $y = 1, \cdots, q$, are the eigenvalues of the matrix **U**. In the case of the standard Potts model, (6) reduces to

$$u^*(y) = \frac{1}{\sqrt{q}} \left[e^K - 1 + q\,\delta(y, 0) \right]. \tag{8}$$

III. DUALITY RELATION FOR THE CORRELATION FUNCTION

Number n sites on the boundary of $\mathcal{L}$ by integers $1, 2, \cdots n$ clockwise as shown in Fig. 1. The probability that the n sites are in respective spin states $x_1, x_2, \cdots, x_n$ is

$$P_n(x_1, x_2, \cdots, x_n) = Z_{x_1 x_2 \cdots x_n}/Z(u), \tag{9}$$

where $Z_{x_1 x_2 \cdots x_n}$ is the *partial* partition function of the chiral Potts model with the n spins in fixed given states. Construct an auxiliary lattice $\mathcal{L}_{\text{aux}}$ from $\mathcal{L}$ by connecting all n boundary sites to an extra site e in the infinite face of $\mathcal{L}$ as shown. These n connecting lines then divide the infinite face of $\mathcal{L}$ into n parts, which we also number by $1, 2, \cdots, n$ as shown in Fig. 1. Let u_{je}, $j = 1, 2, \cdots n$, be the Boltzmann factor of the line connecting sites j to e. Its dual edge then connects sites (residing in faces of $\mathcal{L}$) j and $j + 1$ on $\mathcal{L}_{\text{aux}}^D$ with the Boltzmann factor

$$u_{j+1,j}^*(y_{j+1} - y_j) = \frac{1}{\sqrt{q}} \sum_{x=1}^{q} \omega^{(y_{j+1} - y_j)x} u_{je}(x), \qquad j = 1, \cdots, n \tag{10}$$

where y_j denotes the spin state of the dual site j. Here we have adopted the convention of [3] in orienting the edges. It is also convenient to consider the n-spin correlation [4]

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[1]The definition of the dual Boltzmann factor adopted here differs from that used in [3] by a factor $\sqrt{q}$. The present definition is consistent to the notion of $(u^*)^* = u$.

$$\Gamma_n = q^n P_n(x, x, \cdots, x) - 1, \tag{11}$$

a quantity which vanishes identically if the n spins are completely uncorrelated.

We apply the duality relation (7) to $\mathcal{L}_{\text{aux}}$. Let $Z^*_{y_1 y_2 \cdots y_n}$ be the partial partition function of the Potts model on $\mathcal{L}^D_{\text{aux}}$ with sites $1, 2, \cdots, n$ in respective states $y_1, y_2, \cdots, y_n$. Then, explicitly, (7) reads

$$q^{-(N+1)/2} \sum_{x_1 x_2 \cdots x_n} \sum_{x_e} u_{1e}(x_{1e}) u_{2e}(x_{2e}) \cdots u_{ne}(x_{ne}) Z_{x_1 x_2 \cdots x_n}$$
$$= q^{-(N^*+n-1)/2} \sum_{y_1 y_2 \cdots y_n} u^*_{21}(y_{21}) u^*_{32}(y_{32}) \cdots u^*_{1n}(y_{1n}) Z^*_{y_1 y_2 \cdots y_n}, \tag{12}$$

where $x_{1e} = x_1 - x_e$, $y_{21} = y_2 - y_1$, etc., and we have used the fact that $\mathcal{L}_{\text{aux}}$ has $N + 1$ sites and $\mathcal{L}^D_{\text{aux}}$ has $N^* + n - 1$ sites.[2]

Due to the cyclic nature of the matrix elements, we can replace the arguments $x_{1e}, x_{2e}, \cdots$ on the LHS of (12) by $x_1, x_2, \cdots$. The summation over x_e can then be carried out leading to an overall factor q. Next we eliminate u^* by using (10). Since all u factors in (12) are independent, as a consequence we can equate the coefficients of $u_{1e}(x_1) u_{2e}(x_2) \cdots u_{ne}(x_n)$ in (12). This leads to the duality identity

$$Z_{x_1 x_2 \cdots x_n} = C_n(q) \sum_{y_1 y_2 \cdots y_n} M(x_1, x_2, \cdots, x_n \mid y_1, y_2, \cdots, y_n) Z^*_{y_1 y_2 \cdots y_n}, \tag{13}$$

where

$$C_n(q) = q^{-n+(N-N^*)/2}, \tag{14}$$

and

$$M(x_1, x_2, \cdots, x_n \mid y_1, y_2, \cdots, y_n) = \omega^{x_{n1} y_1 + x_{12} y_2 + \cdots + x_{(n-1)n} y_n} \tag{15}$$

can be interpreted as the elements of a $q^n \times q^n$ matrix $\mathbf{M}$. Eq. (13) is the duality relation for the chiral Potts model, and is a very general expression. The relation assumes a reduced form for the standard Potts model after taking into account the degeneracy of states, and the crux of matter is, of course, to deduce the reduced relation. This is the subject matter of the next section.

Introducing the notation

$$p_{y_1 y_2 \cdots y_n} \equiv Z^*_{y_1 y_2 \cdots y_n} / Z^*_{11 \cdots 1} = q Z^*_{y_1 y_2 \cdots y_n} / Z^{(D)} \tag{16}$$

and combining with (7) and (9), we can rewrite (13) as

$$P_n(x_1, x_2, \cdots, x_n) = q^{-(n+1)} \sum_{y_1 y_2 \cdots y_n} M(x_1, x_2, \cdots, x_{n-1}, x_n \mid y_1, y_2, \cdots, y_n) p_{y_1 y_2 \cdots y_n}. \tag{17}$$

An immediate consequence of (17) is

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[2]Note that the partial partition function $Z^*_{y_1 y_2 \cdots y_n}$ in (12) differs from that in [4,5] by a factor.

$$\Gamma_n = \frac{1}{q} \sum_{y_1 y_2 \cdots y_n} p_{y_1 y_2 \cdots y_n} - 1, \tag{18}$$

a result which has been reported previously [5,12].

To obtain the inverse of (13), we note that, by deleting the $\{1,2\}, \{2,3\}, \cdots$ edges in $\mathcal{L}_{\text{aux}}^D$, we arrive at a lattice $\mathcal{L}^*$ in which $1, 2, \cdots$ of $\mathcal{L}^*$ are its boundary sites. An example of $\mathcal{L}$ and the resulting $\mathcal{L}^*$ is shown in Fig. 2. It is clear that the roles of $\mathcal{L}$ and $\mathcal{L}^*$ are reciprocal, so we can apply (13) to $\mathcal{L}^*$ to obtain an inverse transform. Now $\mathcal{L}^*$ has $\bar{N} = N^* + n - 1$ sites and its dual has $\bar{N}^* = N - n + 1$ sites. As a result, we obtain

$$Z^*_{y_1 y_2 \cdots y_n} = \bar{c}_n(q) \sum_{x_1 x_2 \cdots x_n} M(y_1, y_2, \cdots, y_n \mid x_n, x_1, x_2, \cdots, x_{n-1}) Z_{x_1 x_2 \cdots x_n}, \tag{19}$$

where

$$\bar{c}_n(q) = q^{-n + (\bar{N} - \bar{N}^*)/2} = q^{-1 + (N^* - N)/2}. \tag{20}$$

We have $c_n(q) \bar{c}_n(q) = q^{-(n+1)}$ and note that the sequence $\{x_n, x_1, x_2, \cdots, x_{n-1}\}$ of the indices on the RHS is related to $\{x_1, x_2, \cdots, x_{n-1}, x_n\}$ by a simple cyclic permutation. Upon combining (19) with (13) we obtain the identity

$$\mathbf{M}^2(x_1, x_2, \cdots, x_n \mid x'_1, x'_2, \cdots, x'_n) = q^{n+1} \delta(x_1, x'_2) \delta(x_2, x'_3) \cdots \delta(x_n, x'_1), \tag{21}$$

which we refer to as a reciprocal inversion relation.

IV. THE STANDARD POTTS MODEL

For the standard Potts model the duality relation (13) assumes a reduced form which has been given previously for $n = 2, 3$ [4] and for $n = 4$ [5]. The form for general n has also been conjectured [5]. It has also been shown that the correlation functions satisfy certain sum-rule identities [5]. Here, we reformulate the standard Potts model as an instance of the chiral Potts model and, as we shall see, this leads us to establish the conjecture as well as deduce the general expression for the sum rule identities. For this reason, it is instructive to first demonstrate how (13) reduces to the known results for $n = 2, 3$.

The standard Potts model is characterized by the fact that one needs only to keep track of spin states that are the same. That is to say, one needs to keep track of the partition of the n sites into *blocks*, such that sites in one block are in the same state. This leads to writing the partial partition functions in an expansion of a partially ordered set. Along this line, we writes for $n = 2, 3$,

$$Z_{x_1 x_2} = D_{12} + D_{11} \delta(x_1, x_2)$$
$$Z^*_{y_1 y_2} = D^*_{12} + D^*_{11} \delta(y_1, y_2)$$
$$Z_{x_1 x_2 x_3} = D_{123} + D_{113} \delta(x_1, x_2) + D_{121} \delta(x_1, x_3) + D_{122} \delta(x_2, x_3) + D_{111} \delta(x_1, x_2, x_3)$$
$$Z^*_{y_1 y_2 y_3} = D^*_{123} + D^*_{113} \delta(y_1, y_2) + D^*_{121} \delta(y_1, y_3) + D^*_{122} \delta(y_2, y_3) + D^*_{111} \delta(y_1, y_2, y_3), \tag{22}$$

where $\delta(x_1, x_2, x_3) = \delta(x_1, x_2) \delta(x_2, x_3)$. Here, for $n = 3$, for example, the partition of the 3-set of integers $\{1, 2, 3\}$ includes the 5 elements $\{\{1\}, \{2\}, \{3\}\}$, $\{\{12\}, \{3\}\}$, $\{\{1\}, \{23\}\}$, $\{\{13\}, \{2\}\}$, $\{\{123\}\}$ denoted by subscripts $\{123\}, \{113\}, \{122\}, \{121\}, \{111\}$, respectively.

For $n = 2$, we substitute the first two lines of (22) into (13). Making use of the identity

$$\sum_{y=1}^{q} \omega^{xy} = q\, \delta(x, 0), \tag{23}$$

one arrives at

$$D_{12} + D_{11}\delta(x_1, x_2) = C_2(q) \sum_{y_1 y_2} \omega^{x_{21}y_1 + x_{12}y_2} \left[D_{12}^* + D_{11}^* \delta(y_1, y_2) \right]$$
$$= C_2(q) \left[q^2 D_{12}^* \delta(x_1, x_2) + q D_{11}^* \right]. \tag{24}$$

Since this equation holds for arbitrary x_1 and x_2, the coefficients of corresponding delta functions must be equal, and this leads to the identities

$$D_{12} = C_2(q)\, q D_{11}^*, \qquad D_{11} = C_2(q)\, q^2 D_{12}^*, \tag{25}$$

which can further be converted to relate $Z_{x_1 x_2}$ to Z_{x_1, x_2}^* as follows. First, the second line of (22) gives

$$Z_{12}^* = D_{12}^*, \qquad Z_{11}^* = D_{12}^* + D_{11}^*, \tag{26}$$

from which one obtains the inverse

$$D_{12}^* = Z_{12}^*, \qquad D_{11}^* = Z_{11}^* - Z_{12}^*. \tag{27}$$

Substituting (27) into (25) and combining (9), (7) and the definition (16), one is led to the duality relation for the 2-point correlation function

$$P_2(x_1, x_2) = [D_{12} + D_{11}\delta(x_1, x_2)]/Z$$
$$= \frac{1}{q^2} \left[1 + \left(q\delta(x_1, x_2) - 1 \right) p_{12} \right]. \tag{28}$$

This is the duality relation (17) for $n = 2$. Finally, using (22), (25), and (27), and introducing the row vector $\tilde{\mathbf{z}}_2 = (Z_{11}, Z_{12})$, $\tilde{\mathbf{z}}_2^* = (Z_{11}^*, Z_{12}^*)$, we find

$$\mathbf{z}_2 = q C_2(q) \mathbf{T}_2(q) \cdot \mathbf{z}_2^*, \tag{29}$$

where

$$\mathbf{T}_2(q) = \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix}, \tag{30}$$

satisfying $[\mathbf{T}_2(q)]^2 = q\mathbf{I}_2$, with $\mathbf{I}_2$ being the 2×2 identity matrix.

Similarly for $n = 3$ one obtains in place of (24)

$$D_{123} + D_{113}\delta(x_1, x_2) + D_{121}\delta(x_1, x_3) + D_{122}\delta(x_2, x_3) + D_{111}\delta(x_1, x_2, x_3)$$
$$= C_3(q) \sum_{y_1 y_2 y_3} \omega^{x_{31}y_1 + x_{12}y_2 + x_{23}y_3} \left[D_{123}^* + D_{113}^*\delta(y_1, y_2) + D_{121}^*\delta(y_1, y_3) \right.$$
$$\left. + D_{122}^*\delta(y_2, y_3) + D_{111}^*\delta(y_1, y_2, y_3) \right], \tag{31}$$

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leading to the identities

$$D_{123} = C_3(q)\, qD^*_{111}, \qquad\qquad D_{111} = C_3(q)\, q^3 D^*_{123}$$
$$D_{113} = C_3\, q^2 D^*_{121}, \quad D_{121} = C_3(q)\, q^2 D^*_{122}, \quad D_{122} = C_3(q)\, q^2 D^*_{113}. \qquad (32)$$

Again, we can rewrite (32) in a form relating $Z_{x_1 x_2 x_3}$ to $Z^*_{y_1 y_2 y_3}$. Now the inverse of the last line of (22) is

$$D^*_{123} = Z^*_{123}$$
$$D^*_{113} = Z^*_{113} - Z^*_{123}, \quad D^*_{121} = Z^*_{121} - Z^*_{123}, \quad D^*_{122} = Z^*_{122} - Z^*_{123},$$
$$D^*_{111} = Z^*_{111} - (Z^*_{113} + Z^*_{121} + Z^*_{122}) + 2Z^*_{123}. \qquad (33)$$

Substituting (33) into (32) and using the third line of (22) for $Z_{x_1 x_2 x_3}$, we obtain

$$P_3(x_1, x_2, x_3) = \frac{1}{q^3}\Big[1 - (p_{122} + p_{121} + p_{113}) + 2p_{123} + q(p_{121} - p_{123})\delta(x_1, x_2)$$

$$+ q(p_{122} - p_{123})\delta(x_1, x_3) + q(p_{113} - p_{123})\delta(x_2, x_3) + q^2 p_{123}\delta(x_1, x_2, x_3)\Big]. \quad (34)$$

This is the duality relation (17) for $n = 3$ reported in [4].[3]

Explicitly, introducing the row vector $\tilde{\mathbf{z}}_3 = (Z_{111}, Z_{113}, Z_{121}, Z_{122}, Z_{123})$ and similarly defined $\tilde{\mathbf{z}}^*_3$, we find

$$\mathbf{z}_3 = qC_3(q)\mathbf{T}_3(q) \cdot \mathbf{z}^*_3, \qquad (35)$$

where

$$\mathbf{T}_3(q) = \begin{pmatrix} 1 & q-1 & q-1 & q-1 & (q-1)(q-2) \\ 1 & -1 & q-1 & -1 & -(q-2) \\ 1 & -1 & -1 & q-1 & -(q-2) \\ 1 & q-1 & -1 & -1 & -(q-2) \\ 1 & -1 & -1 & -1 & 2 \end{pmatrix} \qquad (36)$$

satisfying

$$[\mathbf{T}_3(q)]^2 = q^2 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (37)$$

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[3]Note, however, the numbering of $\{y_1, y_2, y_3\}$ as $\{s', s'', s\}$ in [4], resulting in a more symmetric appearance of (34).

V. THE GENERAL ANALYSIS

For general n, we write $Z_{x_1 \cdots x_n}$ as a sum in the form of (22), namely,

$$Z_{x_1 \cdots x_n} = \sum_X D_X \delta(X), \tag{38}$$

where the summation is over all partitions X of n integers $\{1, 2, \cdots, n\}$, and $\delta(X)$ is a product of delta functions associated with blocks in X. For example, we have $\delta(X) = \delta(x_1, x_2)\delta(x_3, x_4)$ for $X = \{1133\}$, and $\delta(X) = \delta(x_1, x_2, x_3) = \delta(x_1, x_2)\delta(x_2, x_3)$ for $X = \{1114\}$. If a block of X contains $m \geq 3$ x's as in $\delta(x_1, x_2, x_3)$ above, we write $\delta(X)$ as a product of $m - 1$ 2-point delta functions $\delta(x_j, x_k)$ with the x's arranged in, say, a clockwise sequence, so the writing of $\delta(X)$ as a product of 2-point delta functions is unique. The number of terms in (38) is b_n, and it has been shown in [5] that b_n is generated by

$$\sum_{n=0}^{\infty} b_n t^n / n! = \exp(e^t - 1). \tag{39}$$

Without the loss of generality we shall assume $q \geq n$ so that all Z_X are physically realized.[4]

The partition sums (22) and (38) can be regarded as a transformation between the Z's and D's. The transformation is that of a partially ordered set and it is known [13] that the inverse is given by the Möbius inversion. Let X, X', Y, etc. denote specific partitions of an n-set. We write $X' \preceq X$ if the block structure of X' is contained in X, namely, X' is a refinement of X [13]. For example, $\{1134\}$ and $\{1234\}$ are contained in $\{1133\}$, while $\{1224\}$ is not. Then, an immediate consequence of (38) is [13]

$$Z_X = \sum_{X' \preceq X} D_{X'}. \tag{40}$$

In addition, the Möbius inversion of (38) is

$$D_X = \sum_{X' \preceq X} \mu(X', X) Z_{X'} \tag{41}$$

with

$$\mu(X', X) = (-1)^{|X'| - |X|} \prod_{\text{blocks} \in X} (n_b - 1)!, \tag{42}$$

where $|X|$ and $|X'|$ are, respectively, the number of blocks in X and X', and n_b the number of blocks of X' that are contained in a block of X. In the examples above, for instance, $X = \{1133\}$ has two blocks, and we have $n_b = \{1, 2\}$ for $X' = \{1134\}$, and $n_b = \{2, 2\}$ for $X' = \{1234\}$, etc. This leads to

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[4]Otherwise the situation $q < |X| \leq n$ can occur, where $|X|$ is the number of blocks of the partition X, and Z_X has no physical meaning. But Z_X, which can be computed for $q \geq |X|$, is a polynomial in q. We shall take Z_X to be the same polynomial for all q including $q < |X|$. With this understanding in mind all our results, including the identities derived below, hold for all q, n.

$$D_{1133} = (-1)^{2-2}(0!)^2 Z_{1133} + (-1)^{3-2}(0!)(1!)Z_{1134} + (-1)^{3-2}(1!)(0!)Z_{1233} + (-1)^{4-2}(1!)^2 Z_{1234}$$
$$= Z_{1133} - Z_{1134} - Z_{1233} + Z_{1234}.$$

$$(43)$$

Other examples are

$$D_{1114} = Z_{1114} - Z_{1134} - Z_{1214} - Z_{1224} + 2Z_{1234}$$
$$D_{1212} = Z_{1212} - Z_{1214} - Z_{1232} + Z_{1234}.$$

$$(44)$$

Substitute the partition sum (38) and a similar expression for $Z^*_{y_1 \cdots y_n}$ into (13), we obtain

$$\sum_X D_X \delta(X) = C_n(q) \sum_Y \sum_{y_1 \cdots y_n} M(x_1, \cdots x_n | y_1, \cdots y_n) D^*_Y \delta(Y).$$

$$(45)$$

It turns out that the analysis is best done graphically. Represent the matrix element $M(x_1, \cdots, x_n | y_1, \cdots, y_n)$ in (13) by a graph G_n shown in Fig. 3, where each node (open circle) denotes a y-summation, and each arc is assigned a label x such that an outgoing arrow (from a node labelled y) carries a factor ω^{-xy} and an incoming arrow a factor ω^{xy}. To each X we construct a "connectivity" Γ_X according to the following prescription: Connect the mid-points of the arcs belonging to each block of X to a common point exterior to G_n. A connectivity is planar if the connecting lines do not intersect; otherwise the connectivity is non-planar. Examples of connectivities for G_4 are shown in Fig. 4. We note that the construction of the connectivities is the same as that introduced in [5]. We have two possibilities.

i) Γ_X is planar. In this case Γ_X divides the region exterior to G_n into *faces*. Regarding all nodes in one face as belonging to a block of another partition Y, and denote this mapping by $X \to Y$. It is clear that Γ_Y is also planar and that $Y \to X$. Therefore the mapping $X \leftrightarrow Y$ is one-one. One has also[5]

$$|X| + |Y| = n + 1.$$

$$(46)$$

It has been shown [14,15] that the number c_n of planar Γ_X, or simply planar X, is generated by

$$\sum_{n=0}^{\infty} c_n t^n = (1 - \sqrt{1 - 4t})/2t.$$

$$(47)$$

With these understandings, we now carry out the y-summations on the RHS of (45). From (23) each y-summation yields a factor $q\, \delta(x, x')$, where x and x' are the labellings of the incoming and outgoing arrows. One can then regard the labellings x as a flow which is conserved at each node. Further, the effect of the delta function $\delta(Y)$ in (45) is to collapse the y-summations of each block of Y into a single summation, resulting in a factor $q\, \delta(x_{\text{in}}, x_{\text{out}})$ where x_{in} is the sum of the incoming flows and x_{out} the sum of outgoing flows. This effect

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[5]For $X = \{\{x_1\}, \{x_2\}, \cdots, \{x_n\}\}$, $|X| = n$, we have $Y = \{\{y_1, y_2, \cdots, y_n\}\}$, $|Y| = 1$. It is also clear that $|Y|$ increases by 1 when $|X|$ decreases by 1. This establishes (46).

can be conveniently visualized by contracting the nodes in question into a single one, and requiring that the flow is conserved at the contracted node. For example, as shown in Fig. 5, the partition $Y = \{123216\}$ of G_6 is contracted into a diagram consisting of a sequence of 3 G_2 "cactus leaves" attached together at two common nodes. Generally, the delta function $\delta(Y)$ contracts G_n into a diagram consisting of cactus leaves $G_m, m < n$, attached together at common nodes. The planar nature of Y then ensures that the resulting diagram is a cactus *tree*. One next carries out the y-summations one-by-one starting from the nodes of the outermost leaves of the tree. The tree structure now ensures that this always leads to a product of 2-point delta functions. In the example shown in Fig. 5, for instance, this leads to

$$\delta(x_5, x_6)\delta(x_4 + x_6, x_1 + x_5)\delta(x_1 + x_3, x_2 + x_4)\delta(x_2, x_3) = \delta(x_5, x_6)\delta(x_4, x_1)\delta(x_2, x_3). \quad (48)$$

But this is precisely the factor $\delta(X)$ appearing in $D_X\delta(X)$ on the LHS of (45), where $X = \{122155\} \leftrightarrow Y$. It is readily verified that, generally for each planar $X \leftrightarrow Y$, to the term $D_X\delta(X)$ on the LHS of (45), the y-summations of the Y term on the RHS yields a factor $D_Y q^{|Y|}\delta(X)$. Now, the indices $\{x_i\}$ in a given $\delta(X)$ are arbitrary. It follows that the coefficients of the two terms must be equal, and this leads to the identity

$$D_X = C_n(q)\, q^{|Y|}D_Y^*, \qquad \text{planar } X \leftrightarrow Y, \quad (49)$$

an equation which generalizes (25) and (32).

ii) Γ_X is non-planar. We now consider the remaining $b_n - c_n$ terms in (45). Now, each remaining terms on the LHS is of the form $D_X\delta(X)$, where $\delta(X)$ is a product of 2-point delta functions $\delta(x_j, x_k)$. For the remaining terms on the RHS we again carry out the y-summations for each Y. However, due to the fact that Y is now non-planar, the process of contraction as described in the above leads to diagrams in which cactus leaves form rings, or circuits. It follows that the y-summations generates a product involving some 4-point delta functions of the type $\delta(x_i + x_j, x_k + x_\ell)$, which cannot be reduced into products of 2-point delta functions as in (48). In other words, the delta functions $\delta(X)$ occurring on the LHS do not appear on the RHS, and vice versa. Now, the the indices $\{x_i\}$ in a given $\delta(X)$ are arbitrary. It follows that, for the equality (45) to hold for all $\{x_i\}$, each of these remaining coefficients must vanish individually. Namely, we must have

$$D_X = D_Y^* = 0, \qquad \text{non} - \text{planar } X, Y. \quad (50)$$

Finally introduce the partially ordered partition sums

$$P_n(x_1, x_2, \cdots, x_n) = \sum_X A_X\delta(X)$$

$$p_{y_1y_2\cdots y_n} = \sum_Y B_Y\delta(Y), \quad (51)$$

where $A_X = D_X/Z$, $B_X = Z_Y^*/Z_{11\cdots1}^*$. Combining (7), (16) and using (46), the identities (49) and (50) now establish the conjecture of [5], which we now state as a
Theorem:

$$A_X = q^{-|X|}B_Y, \qquad \text{for planar } X \leftrightarrow Y,$$

$$= 0, \qquad \text{otherwise.} \quad (52)$$

10

Explicitly, the coefficients B_Y are given in terms of $p_{y_1 y_2 \cdots y_n}$ by the Möbius inversion (41). Thus, the second line in (52) leads to identities relating partial partition functions Z_X. For example, from the second line of (44) and $A_{1212} = D_{1212} = 0$, we obtain

$$Z_{1212} - Z_{1214} - Z_{1232} + Z_{1234} = 0. \tag{53}$$

This is the $n = 4$ identity reported in [5]. Generally, one obtains an identity $A_X = 0$ for each non-planar X. Using these equations one can express all non-planar Z_X in terms of planar ones, a fact reached in [5] through the use of high-temperature expansions.

One can also write down the transformation relating Z_X and Z_Y^*. For general n and planar X one finds by combining (40), (49) and (41),

$$Z_X = C_n(q) \sum_{X' \preceq X} q^{|Y'|} \sum_{Y'' \preceq Y'} \mu(Y'', Y') Z_{Y''}^*, \qquad X' \leftrightarrow Y'. \tag{54}$$

The transformation relating $P_n(x_1, \cdots, x_n)$ to $p_{y_1 \cdots y_n}$ can be written down similarly with the factor $C_n(q)$ replaced by $q^{-(n+1)}$. More explicitly, define a row vectors $\tilde{\mathbf{z}}_n$ and $\tilde{\mathbf{z}}_n^*$ whose elements are the c_n partial partition functions corresponding to planar X, we find

$$\mathbf{z}_n = q C_n(q) \mathbf{T}_n(q) \cdot \mathbf{z}_n^*, \tag{55}$$

where $\mathbf{T}_n(q)$ is a $c_n \times c_n$ matrix satisfying the identity

$$[\mathbf{T}_n(q)]^2 (X, X') = q^{n-1} \delta(x_1, x_2') \delta(x_2, x_3') \cdots \delta(x_n, x_1'). \tag{56}$$

Note that the coefficient q^{n-1} on the RHS differs from that in (21). Expressions of $\mathbf{T}_n(q)$ for $n = 2, 3$ have been given in (30) and (36). The expression for $n = 4$ can be deduced from results reported in [5]. For $n = 5$ there are 42 planar X and we define the row vector

$$\begin{aligned}
\tilde{\mathbf{z}}_5 = \Big(& Z_{11111}, (Z_{11115}, Z_{11141}, Z_{11311}, Z_{12111}, Z_{12222}), (Z_{11144}, Z_{11331}, Z_{12211}, Z_{11333}, Z_{12221}), \\
& (Z_{11145}, Z_{11341}, Z_{12311}, Z_{12333}, Z_{12225}), (Z_{11335}, Z_{12241}, Z_{11344}, Z_{12331}, Z_{12244}), \\
& (Z_{11315}, Z_{12141}, Z_{12311}, Z_{12115}, Z_{12242}), (Z_{11343}, Z_{12321}, Z_{12144}, Z_{12332}, Z_{12215}), \\
& (Z_{12125}, Z_{12142}, Z_{12312}, Z_{12313}, Z_{12323}), (Z_{11345}, Z_{12341}, Z_{12344}, Z_{12335}, Z_{12245}), \\
& (Z_{12145}, Z_{12342}, Z_{12315}, Z_{12343}, Z_{12325}), Z_{12345} \Big).
\end{aligned} \tag{57}$$

Then after some algebras we deduce from (54) the explicit transformation

$$\begin{aligned}
Z_{11111} = \{ & 1 + q_1(1,1,1,1,1) + q_1(1,1,1,1,1) + q_1 q_2(1,1,1,1,1) + q_1 q_2(1,1,1,1,1) \\
& + q_1^2(1,1,1,1,1) + q_1^2(1,1,1,1,1) + q_1 s(1,1,1,1,1) + q_1^2 q_2(1,1,1,1,1) + q_1 q_2 t \}, \\
Z_{11115} = \{ & 1 + (-1, q_1, q_1, q_1, -1) + (-1, q_1, q_1, -1, q_1) - q_2(1, -q_1, -q_1, 1, 1) \\
& - q_2(1, -q_1, 1, -q_1, 1) - q_1(1, -q_1, 1, 1, 1) - q_1(1, -q_1, 1, 1, 1) + s(1, -q_1, 1, 1, 1) \\
& - q_1 q_2(1, 1, 1, 1, 1) - q_2 t \}, \\
Z_{11144} = \{ & 1 + (q_1, -1, q_1, q_1, -1) - (1, 1, -q_1, 1, 1) - q_2(1, 1, -q_1, 1, 1) - q_2(1, 1, 1, 1, 1) \\
& + (-q_1^2, -q_1, -q_1, q_1^2, 1) + (1, -q_1, -q_1, 1, q_1^2) - (p, s, s, p, p) \\
& + q_2(-q_1, 1, q_1 q_2, 1, -q_1) - q_2 u \},
\end{aligned}$$

$$Z_{11145} = \{1 - (1, 1, -q_1, -q_1, 1) - (1, 1, -q_1, 1, 1) + (2, -q_2, q_1q_2, -q_2, 2) - (-2, q_2, q_2, q_2, q_2)$$
$$-(q_1, q_1, q_1, q_1, -1) - (-1, q_1, q_1, -1, q_1) + (h, -s, -s, h, h)$$
$$+(2q_1, q_2, -q_1q_2, q_2, 2q_1) + 2(t+q)\},$$
$$Z_{11335} = \{1 - (1, -q_1, 1, -q_1, 1) - (1, 1, 1, 1, 1) - q_2(1, 1, 1, 1, 1) + (2, -q_2, 2, -q_2, 2)$$
$$+(1, q_1^2, 1, -q_1, -q_1) + (-q_1, 1, -q_1, 1, 1) + (h, -p, h, h, h) - (q_1q_2, q_2, q_2, q_1q_2, 2) + 2u\},$$
$$Z_{11315} = \{1 - (1, 1, 1, -q_1, 1) - (1, -q_1, 1, 1, -q_1) - (-2, q_2, q_2, q_2, q_2) + (-q_2, -q_2, 2, q_1q_2, 2)$$
$$+(1, -q_1, 1, -q_1, 1) + (1, -q_1, 1, -q_1, 1) + (e, -s, h, -s, e) + (2q_1, q_2, 2q_1, 2q_1, q_2) + v\},$$
$$Z_{11343} = \{1 - (1, 1, 1, -q_1, 1) - (-q_1, 1, 1, 1, 1) - (q_2, -2, q_2, q_2, -2) - (-2, -2, q_2, q_2, q_2)$$
$$+(1, -q_1, 1, -q_1, 1) + (1, 1, 1, 1, 1) + (e, h, -p, h, e) - (q_1q_2, 2, -q_2, -q_2, 2) + v + 3q\},$$
$$Z_{11345} = \{1 - (1, 1, 1, -q_1, 1) - (1, 1, 1, 1, 1) + (2, 2, -q_2, -q_2, 2) + (2, 2, 2, -q_2, 2)$$
$$+(1, -q_1, 1, -q_1, 1) + (1, 1, -q_1, 1, 1) + (-5, h, h, h, -5) + (2q_1, -2, q_2, q_2, -2) + 14 - 5q\},$$
$$Z_{12145} = \{1 - (1, 1, 1, 1, 1) - (1, 1, -q_1, 1, 1) + (2, 2, -q_2, 2, 2) + (2, -q_2, 2, 2, -q_2)$$
$$+(1, 1, 1, 1, 1) + (1, -q_1, 1, 1, -q_1) - (q+5, 5, 5, 5, -h) - (2, 2, -q_2, 2, 2) + 14 - 2q\},$$
$$Z_{12345} = \{1 - (1, 1, 1, 1, 1) - (1, 1, 1, 1, 1) + (2, 2, 2, 2, 2) + (2, 2, 2, 2, 2)$$
$$+(1, 1, 1, 1, 1) + (1, 1, 1, 1, 1) - (5, 5, 5, 5, 5) - (2, 2, 2, 2, 2) + 14\}, \tag{58}$$

where $q_m = q - m$, $e = q - 5$, $h = 2q - 5$, $s = q^2 - 5q + 5$, $p = q^2 - 4q + 5$, $t = q^2 - 7q + 7$, $u = q^2 - 5q + 7$, $v = q^2 - 10q + 14$. Here, one reads off elements of $\mathbf{T}_5(q)$ from (58) directly. For example, the second row of $\mathbf{T}_5(q)$ has elements $\{1, (-1, q_1, q_1, q_1, -1), (-1, \cdots), \cdots, -q_2 t\}$. Elements not shown are given by cyclic permutations of the partition indices within each set of parentheses. For example, the third row of $\mathbf{T}_5(q)$ is $\{1, (-1, -1, q_1, q_1, q_1), (q_1, \cdots), \cdots, -q_2 t\}$, where elements within each set of parentheses are obtained by cyclically permuting those in the second row.

VI. THE MULTI-COMPONENT POTTS AND CHIRAL POTTS MODELS

The chiral Potts model can be generalized to more-than-one components. An m-component chiral Potts model [7,8] is a

$$Q = N_1 N_2 \cdots N_m \tag{59}$$

state spin model, where $N_\ell, \ell = 1, 2, \cdots, m$, are positive integers. The interaction matrix $\mathbf{U}$ is a $Q \times Q$ matrix in the form of a direct product of m cyclic matrices, with elements indexed by $U(x_1, \cdots, x_m | x_1', \cdots, x_m')$, $x_\ell, x_\ell' = 1, 2, \cdots, N_\ell$, $\ell = 1, \cdots, m$, satisfying the cyclic property

$$U(x_1, \cdots, x_m \mid x_1', \cdots, x_m') = u(x_1 - x_1', \cdots, x_m - x_m'). \tag{60}$$

The $m = N_1 = N_2 = 2$ version is known as the Ashkin-Teller model [9].

It is convenient to introduce vectors $\mathbf{x} = (x_1, \cdots, x_m)$ and $\mathbf{y} = (y_1, \cdots, y_m)$ and their scalar product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 / N_1 + \cdots + x_m y_m / N_m, \qquad x_\ell, y_\ell = 1, \cdots, N_\ell. \tag{61}$$

12

Then the dual model has the Boltzmann factor

$$u^*(\mathbf{y}) = \frac{1}{\sqrt{Q}} \sum_{x_1=1}^{N_1} \cdots \sum_{x_m=1}^{N_m} e^{2\pi i \mathbf{x} \cdot \mathbf{y}} \, u(\mathbf{x}), \tag{62}$$

and the duality relation (7) now reads

$$Q^{-N/2} Z[u(\mathbf{x})] = Q^{-N^*/2} Z^{(D)}[u^*(\mathbf{y})]. \tag{63}$$

Here, again, $\sqrt{Q} u^*(\mathbf{y})$ are the eigenvalues of the interaction matrix $\mathbf{U}$.

Proceeding in a similar fashion as before, we arrive at the following duality relation in place of (13),

$$Z_{\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n} = C_n(Q) \sum_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} M(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \mid \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) Z^*_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n}. \tag{64}$$

Here, $\mathbf{M}$ is a $Q^n \times Q^n$ matrix with elements

$$M(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \mid \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) = \exp\Big[2\pi i (\mathbf{x}_{n1} \cdot \mathbf{y}_1 + \mathbf{x}_{12} \cdot \mathbf{y}_2 + \cdots + \mathbf{x}_{(n-1)n} \cdot \mathbf{y}_n) \Big], \tag{65}$$

satisfying the reciprocal inversion relation

$$\mathbf{M}^2(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \mid \mathbf{x}_1', \mathbf{x}_2', \cdots, \mathbf{x}_n') = Q^{n+1} \delta(\mathbf{x}_1, \mathbf{x}_2') \delta(\mathbf{x}_2, \mathbf{x}_3') \cdots \delta(\mathbf{x}_n, \mathbf{x}_1'). \tag{66}$$

Defining the n-point correlation functions

$$P_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = Z_{\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n} / Z(u),$$
$$p_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} = Q Z^*_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} / Z^{(D)}(u^*), \tag{67}$$

then (64) becomes

$$P_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = Q^{-(n+1)} \sum_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} M(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \mid \mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) p_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n}. \tag{68}$$

This is the most general correlation duality relation for the m-component chiral Potts model. An immediate consequence of (68) is, in analogous to (11) and (18), the following expression for the n-point correlation

$$\Gamma_n \equiv Q^n P_n(\mathbf{x}, \mathbf{x}, \cdots, \mathbf{x}) - 1 = Q^{-1} \sum_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} p_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} - 1. \tag{69}$$

The m-component Potts model: The m-component chiral Potts model reduces to an m-component (standard) Potts model when the interaction (60) assumes the form (5) for each component, namely,

$$u(x_1, \cdots, x_m) = \prod_{\ell=1}^m \Big[1 + (e^{K_\ell} - 1) \delta_{N_\ell}(x_\ell, 0) \Big], \quad x_\ell = 1, 2, \cdots, N_\ell, \tag{70}$$

where

13

$$\delta_{N_\ell}(x, x') = 1, \qquad x = x' \pmod{N_\ell},$$
$$= 0, \qquad x \neq x' \pmod{N_\ell}. \tag{71}$$

The $m = 2$ version is known as the (N_1, N_2) model [10].

Now each $\mathbf{x}_i$ in (68) is an m-component vector. Let its components be $x_{i\ell} = 1, \cdots, N_\ell$, $i = 1, \cdots, n$; $\ell = 1, \cdots, m$, and let X_ℓ denote the partition of the n integers $1, 2, \cdots, n$ dictated by $\{x_{i\ell}, \ i = 1, \cdots, n\}$. Then, in analogous to (51), we write the partially ordered mappings

$$P_n(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = \sum_{X_\ell} A_{X_1 X_2 \cdots X_m} \delta_{N_1}(X_1) \delta_{N_2}(X_2) \cdots \delta_{N_m}(X_m),$$
$$p_{\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_n} = \sum_{Y_\ell} B_{Y_1 Y_2 \cdots Y_m} \delta_{N_1}(Y_1) \delta_{N_2}(Y_2) \cdots \delta_{N_m}(Y_m). \tag{72}$$

With these notations, we have the following corollary of our Theorem:

Corollary:

For the m-component Potts model whose correlation functions are (72), we have

$$A_{X_1 X_2 \cdots X_m} = \prod_{\ell=1}^{m} N_\ell^{-|X_\ell|} B_{Y_1 Y_2 \cdots Y_m}, \qquad \text{for planar } X_\ell \leftrightarrow Y_\ell,$$
$$= 0, \qquad \text{otherwise.} \tag{73}$$

Here, explicitly, by using

$$Z_{X_1 X_2 \cdots X_m}/Z = \sum_{X'_\ell \preceq X_\ell} A_{X'_1 X'_2 \cdots X'_m}$$
$$B_{Y_1 Y_2 \cdots Y_m} = \sum_{Y'_\ell \preceq Y_\ell} p_{Y'_1 Y'_2 \cdots Y'_m} \prod_{\ell=1}^{m} \mu(Y'_\ell, Y_\ell), \tag{74}$$

in analogous to (54) one can reduce the correlation duality relation (64) to

$$Z_{X_1 X_2 \cdots X_m} = C_n(Q) \sum_{X'_\ell \preceq X_\ell} \sum_{Y''_\ell \preceq Y'_\ell} Z^*_{Y''_1 Y''_2 \cdots Y''_m}$$
$$\times \prod_{\ell=1}^{m} N_\ell^{|Y'_\ell|} \mu(Y''_\ell, Y'_\ell), \qquad X'_\ell \leftrightarrow Y'_\ell. \tag{75}$$

This is the desired duality relation for the m-component Potts model. Now there are $(c_n)^m$ partial partition functions Z and Z^*. Considering Z and Z^* as tensors $\mathbf{Z}_m$ and $\mathbf{Z}^*_m$ of rank m, one can rewrite (75) more compactly as

$$\mathbf{Z}_m = QC_n(Q)\Big[\mathbf{T}_n(N_1) \otimes \cdots \otimes \mathbf{T}_n(N_m)\Big] \cdot \mathbf{Z}^*_m, \tag{76}$$

where the tensor products are over m $c_n \times c_n$ matrices $\mathbf{T}_n(q)$ defined in (55). The last expression generalizes the $m = 2$ results for the (N_1, N_2) model reported in [11].

VII. SUMMARY AND ACKNOWLEDGEMENT

We have considered the chiral Potts model and obtained the duality relation for its correlation functions of n sites on the boundary of a planar lattice. The result is given by (13) and (17). By specializing this result to the standard Potts model, we establish a recent conjecture of [5] on the correlation duality which we now state in (52) as a Theorem. The explicit duality relation relating the partial partition functions is given in (54), with the $n = 2, 3, 5$ expressions explicitly worked out. The formulation is next extended to the multi-component chiral Potts model, leading to the correlation duality relations (64) and (68). Again, specializing the results to the m-component Potts model, we obtain the Corollary (73) and the correlation duality relations (75) and (76).

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Figure captions

Fig. 1. A planar graph $\mathcal{L}$ and n sites $i, j, \ldots m, \ell$ on the boundary.

Fig. 2. Reciprocal graphs $\mathcal{L}$ (solid lines) and $\mathcal{L}^*$ (broken lines).

Fig. 3. Graphical representation G_n of the matrix element (65).

Fig. 4. Examples of connectivities on G_4. (a) $X = \{1133\}$. (b) $X = \{1114\}$. (c) $X = \{1212\}$.

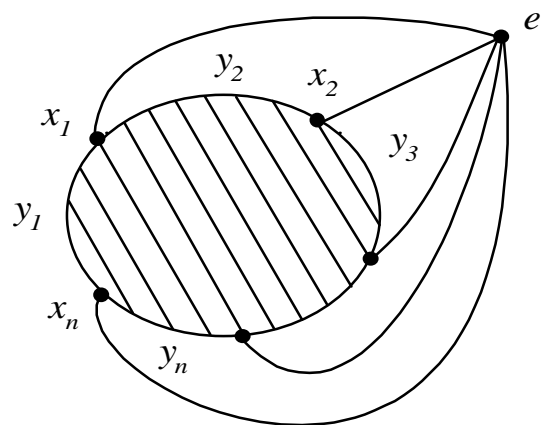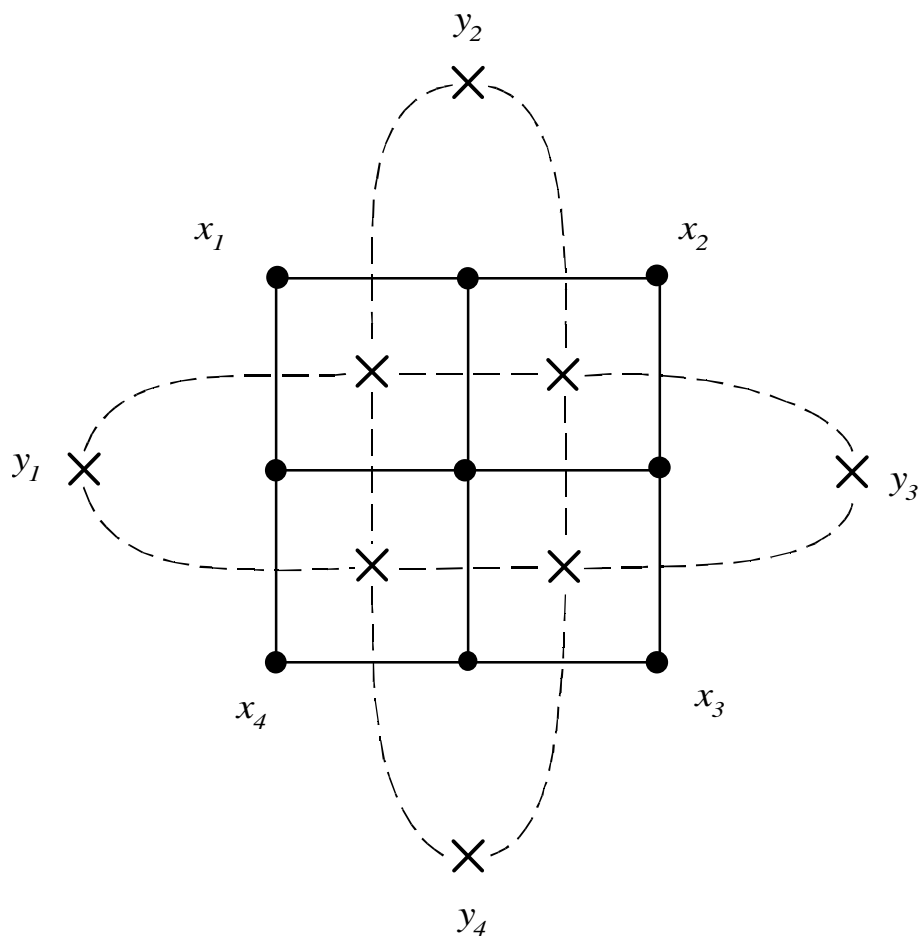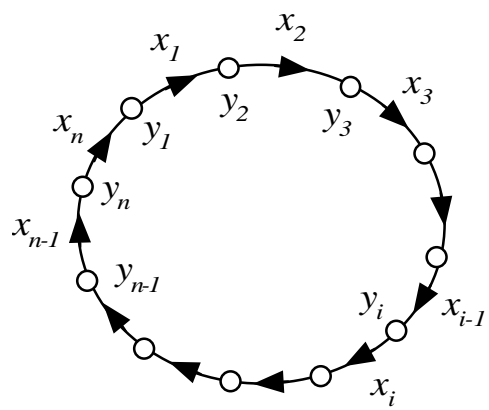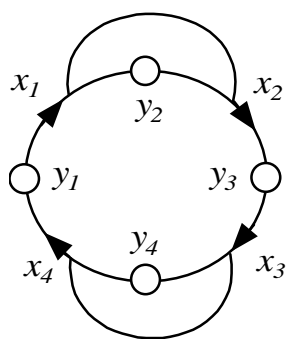Fig. 5. The contraction of the the $Y = \{123216\}$ graph of G_6 into a cactus tree.

Fig. 1
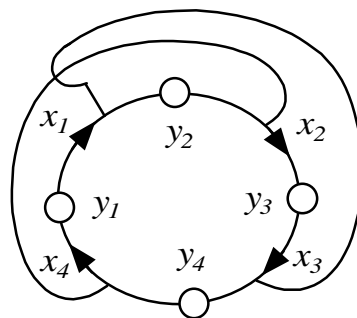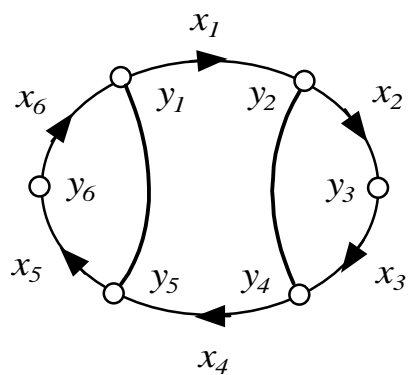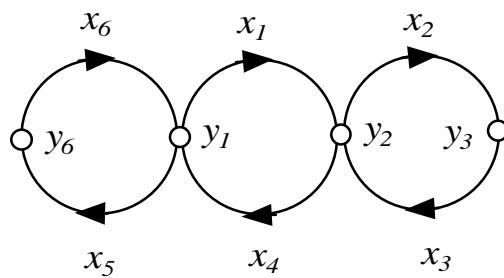
Fig. 2

Fig. 3

(a)   (b)   (c)

Fig. 4

(a)

(b)

Fig. 5